Rijksuniversiteit Groningen
Statistiek

Herentamen

NB. This exam consists of 3 questions with a total number of points: 100.

1. **Maximum likelihood.** Let $X_1, ..., X_n$ be independently exponentially distributed with parameter θ, i.e.

$$p_{X_i}(x) = \frac{e^{-x/\theta}}{\theta}, \quad x > 0$$

   (a) Derive the *full* loglikelihood of θ. (**5 Marks**)
   (b) Find the maximum likelihood estimate of θ? (**10 Marks**)
   (c) Find the Cramer-Rao lower-bound for an *unbiased* estimate of θ. (**10 Marks**)
   (d) Is the MLE unbiased and does it attain the Cramer-Rao lowerbound, using the fact that the mean and variance of X_i are θ and θ^2 respectively? (**10 Marks**)
   (e) If $n = 16$ and $\bar{X} = 2$, then find the approximate 95% confidence interval for θ based on approximate normality of the maximum likelihood estimator. Use the fact that the 97.5% standard normal quantile is $z_{0.975} = 1.96$. (**10 Marks**)
   (f) Find a sufficient statistic for θ. (**10 Marks**)

2. **Rao-Blackwell.** The Rao-Blackwell theorem is given as follows: if $\hat{\theta}$ is an unbiased estimate of θ and T is a sufficient statistic of θ, then

$$\hat{\theta}^* = E(\hat{\theta}|T)$$

   is an unbiased estimate of θ with

$$V(\hat{\theta}^*) \leq V(\hat{\theta})$$

   (a) Prove that $\hat{\theta}^*$ is unbiased. (**5 Marks**)
   (b) Prove that $V(\hat{\theta}^*) \leq V(\hat{\theta})$. (**10 Marks**)

   (Third question on following page.)

3. **Hypothesis testing.** The life span of simple organisms is thought to be modelled well by means of the exponential distribution. A particular organism, the magfly, is thought to have a one day life span. However, by genetic selection a scientist has found a strain with an apparently longer life span. She samples 16 magflies with an average life span of 2 days. The scientist would like to know whether this strain has a significantly different life span compared to the normal magfly with an average life span of only 1 day.

   (a) Write down the random variables, their distributions, the null and alternative hypotheses in this case. **(10 Marks)**

   (b) The likelihood ratio test is defined as

   $$\Lambda = \frac{\max_{\theta \in H_0} L_X(\theta \mid X_1, \ldots, X_{16})}{\max_{\theta \in H_0 \bigcup H_1} L_X(\theta \mid X_1, \ldots, X_{16})}$$

   Derive the likelihood ratio statistic in this case. **(10 Marks)**

   (c) Test whether the new strain found by the scientist has an average life span that is different from 1. By approximation, we can use the fact that

   $$-2\log(\Lambda) \sim \chi_q^2$$

   where q is the difference between the number of free parameters under H_0 and H_1. In your calculations, use the fact that $\log 2 \approx 0.7$ as well as the values from the following table:

df	$\chi^2_{0.05}$	$\chi^2_{0.95}$
3	0.35	7.81
4	0.71	9.49
5	1.15	11.07
99	77.05	123.23
100	77.93	124.34

   (10 Marks)